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Intrinsic Equations for the Nonlinear Dynamics of Space Beams

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Equations are developed for the fully intrinsic nonlinear dynamics of a spatially curved, pretwisted beam. Direct deformation measures are used as variables. Large deformations can be accommodated. The Bernoulli–Euler assumption is introduced for the gross dynamics but not for the constitutive relations. The latter may also be inelastic and time dependent. Initial and end conditions (force and kinematic types) are presented. Specific equations are also given for the planar case and for the linear case. Some of the merits and drawbacks of this approach are discussed. The extension to a continuum is also briefly discussed.

Introduction

THIS paper is concerned with the formulation of the equations for the fully intrinsic nonlinear dynamics of a spatially curved and pretwisted beam (space beam). The equations are constructed in terms of the following direct deformation measures: extension λ, curvatures κ_i of the beam reference line, and the local angular velocities α_i ($i = 1, 2, 3$) of the deforming material base vectors along the line. Position, displacements, rotations, and linear velocities are excluded from the formulation. Any data or end conditions should be specified in terms of the deformed beam directions.

Vector quantities that describe the kinematics and dynamics of the beam are utilized in the process of deriving the intrinsic equations. These are defined in an inertial frame. The vector quantities do not appear in the final form of the equations, but the implicit presence of the inertial frame is expressed through the derivation process. At the initial state, the beam is assumed to be oriented in the inertial frame and its basic quantities, including position, base vectors, dynamic vectors, and velocities, are expressed in this frame. Once the quantities that are needed for the integration process are extracted, no further explicit use of the frame is made, as long as the process is restricted to the calculation of the direct deformation variables. However, if position (displacements), velocities, and rotations are important objectives of the analysis, then the external frame components should be used in the further integration process. In such cases the intrinsic method loses some of its advantages and other formulations may be more useful.

From a geometrical point of view, the formulation can accept very large deformations and a wide range of initial shapes (defined by spatial distributions of curvatures and pretwist). It is required, however, that constitutive algorithms be available for the specified deformation and shape domains. The Bernoulli–Euler (BE) assumption is used for the gross dynamics of the beam but is not necessary for the constitutive relations. It may further restrict the ranges of acceptable deformations, initial shapes, materials, and beam cross sections. More extensive articles that discuss the problems associated with constitutive modeling appear in literature.[1,2] At present, the equations are developed for symmetrical cross sections. Inertial, geometrical, and constitutive asymmetries can, however, be incorporated without difficulty.

In the statics of rods, plates, and shells, the use of direct deformation variables (extensions and changes of curvature) in lieu of displacements and rotations is termed an intrinsic approach. In shear deformation theories, shearing strains may also be added. The nonlinear intrinsic statics of space beams was initially developed in the 19th century by Clebsch and Kirchhoff. It was summarized by Love.[3] Further developments included incorporation of shear deformations, warping effects, and more advanced constitutive relations. Many studies exist. For typical references see Antman,[4] Reissner,[5] and Danielson and Hodges.[6] In the classical BE case, the equilibrium and constitutive equations can be reduced to four differential equations in the three curvature κ_i and extension λ. An important requirement is that the data (loads and end conditions) be expressible in terms of the deformed directions. It is also desirable that the problem be statically determinate. The static planar case has been exhaustively discussed and many solutions exist. It is important to realize that the rotation is related, in this case, directly to the deformation variables. In fact, the elastica problem can also be regarded as intrinsic.

The extension of the analysis to include dynamic terms opens up several possibilities for the representation of those terms, and several authors have used different approaches. Simo[7] and Simo and Vu-Quoc[8] have used an inertial frame for their development, utilizing the fact that the dynamic terms have their simplest representation in such a frame. Hodges[9] utilizes a known moving reference frame for his analysis. The latter is set, of course, in an inertial frame. The use of floating reference frames was discussed by several authors. These are reference frames that are attached to specific points (not necessarily material) in the deforming body. They move and rotate with it, and the deformation is measured with respect to the frame. The advantage lies in the elimination of the large overall motion from the analysis, but the effects of the induced accelerations must be included. (See, for example, McDonough,[10] Canavin and Likins,[11] and Laskin et al.[12])

In a more recent paper, Davis and Hirschorn[13] presented a simplified large deformation dynamics model that is based on curvatures, angular velocities, and linear velocities as variables (its appearance coincided with the present author's paper[14]). Their model was cast in a mixed vector-scalar form, utilizing an inertial frame as reference. The main simplifications included inextensionality, an initially straight beam, no distributed loads, and simple, linearly elastic uncoupled constitutive relations (which possessed some inaccuracies, including an incorrect torsional rigidity coefficient). Their model shares with the present model some features, namely, the use of curvatures and angular velocities as some of its deformation variables and the use of compatibility equations. It does, however, retain the components of the linear velocity as major field variables, thereby following a nonintrinsic direction, as will be discussed later.

Another approach to space beam dynamics was adopted by Green and Laws,[15] who developed equations of motion starting from the dynamics of a Cosserat space line with two directors. Kinematics and thermodynamics were included in the formulation. Their general theory was then specialized to the case of elastic rods. The outcome was nine equations of motion and constitution. The acceleration components were retained in the analysis as additional variables. Further developments in the Cosserat line theory were made by Naghdi and Rubin,[16] who considered several types of constitutive constraints in their analysis. This approach awaits further development and practical applications.

The intrinsic dynamic approach complements the other approaches and processes several advantages compared with these

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methods. Its major drawback lies in the fact that it is not very useful in the rather common case of nonuniform loading or data defined with respect to external reference frames. The merits and the demerits of the method, as well as the classes of problems for which it is useful, will be briefly discussed in the sequel.

The use of intrinsic methods in dynamic problems requires clarification: a pure intrinsic dynamic approach should, seemingly, utilize only the extension and curvatures as field variables. Such an approach, which was moderately successful in the case of doubly curved shells,[18] was not attempted here. Any other intrinsic dynamic method must incorporate either angular velocities or linear velocities (or both). However, in the approach of this paper, an acceptable candidate must produce (or necessitate the coexistence of) deformations to have intrinsic properties. This criterion rules out linear velocities (in fact, very large linear velocities can take place without deformations) but accepts angular velocities. Thus, essentially, the present intrinsic dynamic method utilizes the extension and curvatures (or corresponding stress resultants) as main variables and the angular velocities as auxiliary variables.

The present author has been active in the past in the field of intrinsic dynamics of shells.[17–21] In some sense, this paper may be regarded as an application of the methods developed in shell analysis to the case of nonlinear beam dynamics. Important features that are carried over include the capability to accommodate a wide range of kinematic constraint conditions, the elimination of uniform time-dependent loads and accelerations from the field equations, the inclusion of inelastic and time-dependent constitutive relations, and the setting of the problem in a convenient rate form. For other features, see the discussion in the sequel. Here, as in the static case, the planar problem has been treated more extensively. (See, for example, Simo and Vu-Quoc[22] for the use of inertial frames and Libai[17] for a treatment of the fully intrinsic dynamic case.)

The basic kinematics and equations of motion for space beams are developed and/or appear in the referenced literature. Free use will be made of the results. Some derivations are included to emphasize specific points, and the notation will be adjusted as needed. In the paper, the differential equations for fully intrinsic beam dynamics are developed from the equations of motion and space–time compatibility. They are set up in a time-rate form that is suitable for time integration schemes. Constitutive relations and appropriate end conditions are then presented. Some special cases are also briefly discussed. The merits and demerits of this approach and types of problems for which it is useful are put forth and analyzed.

Geometry and Kinematics

Let a deforming space beam be set in a three-dimensional inertial frame $\mathcal{T}$. All vectors and vectorial operations associated with the beam are defined in $\mathcal{T}$.

Let $\boldsymbol{R}^{(0)}(s)$ denote the position vector, in $\mathcal{T}$, of the centroidal line of the space beam with arclength s at an initial time $t = t_0$. Let $\boldsymbol{u}_i^{(0)}(s)$ $(i = 1, 2, 3)$ denote a right-handed triad of mutually perpendicular unit vectors at any point P on the line, with $\boldsymbol{u}_3^{(0)}(s) = \boldsymbol{R}_{,s}^{(0)}$ being the unit tangent to the line at P. Take r_α $(\alpha = 1, 2)$ as arclength coordinate along straight lines going through $\boldsymbol{u}_\alpha^{(0)}(s)$ at P. The r_α plane cuts the beam along a closed planar curve that is defined to be the beam cross section, and the $\boldsymbol{u}_\alpha^{(0)}$ are taken to be the principal inertia directions of the thin material slice with mass center at P. In general, the directions of $\boldsymbol{u}_i^{(0)}$ vary along the beam but are assumed to be known.

At times $t > t_0$, the material line $\boldsymbol{R}^{(0)}$ deforms into a line $\boldsymbol{R}(s, t)$ (the current line). Here, s is a Lagrangian variable, $\bar{s}$ is arclength, and $\lambda = d\bar{s}/ds$ is the stretch. A unit tangent to $\boldsymbol{R}$ is $\boldsymbol{u}_3(s, t) = \lambda^{-1}\boldsymbol{R}_{,s}(s, t)$, acting at the material image P' of P. The deformed material image of the r_α plane is not planar due to many effects such as shearing deformations, cross-sectional distortions, warping, anisotropy, pretwist, etc. The BE assumption ignores these effects and assumes that the material image of the triad $\boldsymbol{u}_i^{(0)}$ is a mutually orthogonal rotated unit triad $\boldsymbol{u}_i(s, t)$, such that $\boldsymbol{u}_3$ is tangent to $\boldsymbol{R}$ and $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are in the principal directions of the undistorted deformed cross sections. For further discussion, see Love[3] and Rosen.[1] Some of the effects are more important for the constitutive relations than for the gross dynamics of the beam. The approach adopted here emphasizes this point by adopting the BE assumption for the gross dynamics of the beam and, at the same time, allowing for more general constitutive relations. Similar approaches are common in many variants of beam, plate, and shell statics, where a higher level of precision is required from the kinematics than from constitutive relations. This extends into dynamics too.[18,19]

For simplicity, the cross sections of the beam will be assumed to have sufficient symmetry so as to warrant the assumption of the coincidence of the inertial, elastic, and centroidal axes. Slight deviations due to curvature effects are not important. More significant deviations lead to additional coupling that should be taken into account. This can be done within the scope of the present approach but will not be pursued in this paper.

The curvature vector $\boldsymbol{\kappa} = \kappa_i \boldsymbol{u}_i$ and angular velocity vector $\boldsymbol{G} = \alpha_i \boldsymbol{u}_i$ are defined as the space and time derivatives of the triad $\boldsymbol{u}_i(s, t)$ as follows:

$$\boldsymbol{u}_{i,s} = \boldsymbol{\kappa} \times \boldsymbol{u}_i = -e_{ijk}\kappa_j \boldsymbol{u}_k \tag{1}$$

$$\boldsymbol{u}_{i,t} = \boldsymbol{G} \times \boldsymbol{u}_i = -e_{ijk}\alpha_j \boldsymbol{u}_k \tag{2}$$

where e_{ijk} is the three-space permutation symbol and the convention of summing over repeated indices is adopted. Both vectors are widely used in the statics and dynamics of space beams,[7,8] so that details need not be repeated. The use of s (rather than $\bar{s}$) is convenient for time and space separation. Note that, from a geometrical point of view, $\bar{\boldsymbol{\kappa}} = \lambda^{-1}\boldsymbol{\kappa}$ is the true curvature.

Three-dimensional vector functions $\boldsymbol{B}(s, t) = B_i \boldsymbol{u}_i$ may be defined along the beam. Their spatial and temporal derivatives must incorporate the differentiation of the triad itself. Using the aforementioned the expressions for these derivatives bear a formal similarity to those of tensor derivatives, as follows:

$$\boldsymbol{B}_{,s} = B_i|_s \boldsymbol{u}_i; \qquad \boldsymbol{B}_{,t} = B_i|_t \boldsymbol{u}_i \tag{3}$$

where the bar symbol denotes covariant differentiation

$$B_i|_s = B_{i,s} - e_{ijk}B_j\kappa_k \qquad \boldsymbol{B}_i|_t = B_{i,t} - e_{ijk}B_j\alpha_k \tag{4}$$

The κ_i and α_i are interrelated by the equations of space–time compatibility that result from the equality of the second mixed derivatives of $\boldsymbol{B}$: $\boldsymbol{B}_{,st} = \boldsymbol{B}_{,ts}$. The resulting equations are (see Appendix A)

$$\kappa_{i,t} - \alpha_{i,s} = e_{ijk}\kappa_j\alpha_k \tag{5}$$

These equations are useful in dynamic intrinsic analysis. They replace the more common kinematic relations (displacement-strain-curvature change) of the displacement formulations. Some additional relations that stem from the preceding are

$$\boldsymbol{\kappa}_{,t} = \alpha_{i,s}\boldsymbol{u}_i \qquad \boldsymbol{\kappa}_{,s} = \kappa_{i,s}\boldsymbol{u}_i \tag{6}$$

$$\boldsymbol{G}_{,t} = \alpha_{i,t}\boldsymbol{u}_i; \qquad \boldsymbol{G}_{,s} = \kappa_{i,t}\boldsymbol{u}_i \tag{6a}$$

$$\boldsymbol{\kappa}_{,t} - \boldsymbol{G}_{,s} = \boldsymbol{G} \times \boldsymbol{\kappa} \tag{6b}$$

Accelerations and Acceleration Gradients

An essential requirement of the dynamic intrinsic formulations is to be able to express the acceleration terms in the equations of motion in terms of the deformation variables κ_i, α_i, and λ. The translational acceleration vector $\boldsymbol{A} = \boldsymbol{R}_{,tt}$ cannot be brought to intrinsic form, but its spatial gradient

$$\boldsymbol{A}_{,s} = (\lambda\boldsymbol{u}_3)_{,tt} = A_i|_s \boldsymbol{u}_i \tag{7}$$

can be expressed in terms of the direct deformation variables and will be used as the basic kinematical quantity in a new representation of the equations of motion. The simple calculation using the formulas presented earlier yields a general expression for the second time derivatives of $\boldsymbol{u}_i$:

$$(\lambda\boldsymbol{u}_i)_{,tt} = \left[\lambda_{,tt}\delta_{ik} - (1/\lambda)e_{ijk}\left(\lambda^2\alpha_j\right)_{,t} + \lambda e_{pij}e_{prk}\alpha_j\alpha_r\right]\boldsymbol{u}_k \tag{8}$$

where δ_{ik} is the Kronecker delta symbol. Putting $i = 3$, the result for $A_i|_s$ is

$$A_i|_s = \lambda_{,tt}\delta_{3i} - (1/\lambda)e_{3ji}\left(\lambda^2\alpha_j\right)_{,t} + \lambda\beta_i \tag{9}$$

where

$$\beta_1 = \alpha_3\alpha_1; \qquad \beta_2 = \alpha_3\alpha_2; \qquad \beta_3 = -\alpha_1^2 - \alpha_2^2 \tag{9a}$$

A useful expression for the calculation of the angular acceleration is

$$\boldsymbol{u}_m \times \boldsymbol{u}_{i,tt} = e_{mkq}(e_{ijk}\alpha_{j,t} + e_{pij}e_{prk}\alpha_j\alpha_r)\boldsymbol{u}_q$$

In particular, setting $m = i$,

$$\boldsymbol{u}_m \times \boldsymbol{u}_{m,tt} = (\delta_{jk}\alpha_{i,t} + e_{ijk}\alpha_i\alpha_j)\boldsymbol{u}_k \qquad (k \neq m) \tag{10}$$

In the preceding equation, the left side is not summed and the summation of k on the right side extends over $k \neq m$ only.

Equations of Motion and Transformation to Intrinsic Form

The equations of motion follow from the balance of linear and angular momenta. The linear momentum equation yields the standard form,

$$\rho\boldsymbol{R}_{,tt} = \rho\boldsymbol{A} = \boldsymbol{F}_{,s} + \boldsymbol{q} \tag{11}$$

where $\boldsymbol{F} = F_i\boldsymbol{u}_i$ is the cross-sectional force resultant vector, $\boldsymbol{q} = q_i\boldsymbol{u}_i$ is the loading (per unit undeformed length), and ρ is the mass per unit undeformed length.

The component form of the equation is

$$\rho A_i = F_i|_s + q_i \tag{12}$$

As explained before, this equation must be differentiated so that the acceleration be expressible in intrinsic form. This yields

$$\boldsymbol{A}_{,s} = [(1/\rho)(\boldsymbol{F}_{,s} + \boldsymbol{q})]_{,s} \tag{13}$$

Use of the kinematics yields the final component form of the transformed equations:

$$\begin{aligned} A_k|_s &= \lambda_{,tt}\delta_{3k} - (1/\lambda)e_{3jk}\left(\lambda^2\alpha_j\right)_{,t} + \lambda\beta_k \\ &= [(1/\rho)F_k|_s]|_s + [(1/\rho)q_k]|_s \end{aligned} \tag{14}$$

Note that if $(1/\rho)\boldsymbol{q}$ and $\boldsymbol{A}$ are uniform along the beam (i.e., are functions of time only), then they drop out of the transformed equations. This includes the important case of gravity loading.

The vector form of the equation of angular motion comes out of the equation for the balance of angular momentum. It has the standard form

$$\Omega = \lambda\boldsymbol{u}_3 \times \boldsymbol{F} + \boldsymbol{M}_{,s} + \boldsymbol{m} \tag{15}$$

where $\boldsymbol{M} = M_i\boldsymbol{u}_i$ is the cross-sectional moment-resultant vector, with components turning around the corresponding axes; $\boldsymbol{m} = m_i\boldsymbol{u}_i$ is the moment loading (per unit undeformed length); and $\Omega = \Omega_i\boldsymbol{u}_i$ is the time rate of the angular momentum Γ, per unit undeformed length (with respect to the centroidal mass line).

From the definition of Γ, it follows that

$$\Gamma = \iint_S \mu\boldsymbol{r} \times \boldsymbol{r}_{,t}\,\mathrm{d}S \tag{16}$$

where μ is the mass per unit cross-sectional area, per unit undeformed length, and $\boldsymbol{r} = r_1\boldsymbol{u}_1 + r_2\boldsymbol{u}_2$ is the cross-sectional position vector. Hence,

$$\Omega = \iint_S \mu(r_1\boldsymbol{u}_1 + r_2\boldsymbol{u}_2) \times (r_1\boldsymbol{u}_{1,tt} + r_2\boldsymbol{u}_{2,tt})\,\mathrm{d}S \tag{17}$$

If, as assumed, $\boldsymbol{u}_i$ are principal directions, the mixed terms vanish, resulting in

$$\Omega = I_2\boldsymbol{u}_1 \times \boldsymbol{u}_{1,tt} + I_1\boldsymbol{u}_2 \times \boldsymbol{u}_{2,tt} \tag{18}$$

where

$$I_1 = \iint_S \mu r_2^2\,\mathrm{d}S \qquad I_2 = \iint_S \mu r_1^2\,\mathrm{d}S \tag{18a}$$

are the principal mass moments of inertia. Use of previous kinematic results yields the detailed expressions

$$\Omega_1 = I_1(\alpha_{1,t} + \alpha_2\alpha_3) \tag{18b}$$

$$\Omega_2 = I_2(\alpha_{2,t} + \alpha_1\alpha_3) \tag{18c}$$

$$\Omega_3 = (I_2 + I_1)\alpha_{3,t} + (I_2 - I_1)\alpha_1\alpha_2 \tag{18d}$$

The component form of the angular momentum equations becomes

$$\Omega_i = M_i|_s - \lambda e_{3ij}F_j + m_i \tag{19}$$

As is well known, the terms Ω_1 and Ω_2 may be dropped within the BE approximation. This leads to the final form for the balance of angular momenta:

$$\delta_{3k}\Omega_3 - \lambda e_{3jk}F_j = M_k|_s + m_k \tag{20}$$

Constitutive Relations

Neither development nor in-depth discussion of constitutive relations are included in this paper. The existence of constitutive algorithms of the general form

$$\tilde{M}_i(\lambda, \kappa_i, \ldots) \Rightarrow= M_i \tag{21a}$$

$$\tilde{F}_3(\lambda, \kappa_i, \ldots) \Rightarrow= F_3 \tag{21b}$$

is postulated. Here, $\tilde{M}_i$ and $\tilde{F}_3$ are any processes or equations that transform the kinematical variables λ and κ_i (and possibly their time and space derivatives) into moments M_i and force F_3. The material need not be elastic, and the algorithm may also be numerical—as is the case in some elastoplastic analyses. Viscoelastic effects can be introduced via the time derivatives of λ and κ_i.

The most common relations are those of the ordinary approximate theory[3] that can be used in cases of elastic slender beams of relatively compact cross sections and weak initial curvatures. These are

$$\tilde{M}_i = D_i\left[\kappa_i - \kappa_i^{(0)}\right] \qquad \tilde{F}_3 = C(\lambda - 1) \tag{21c}$$

where D_i and C are material-geometrical constants. It should be stressed, however, that in the general case the effects of initial curvatures and pretwist cannot be neglected. The useful approximation of inextensionality is effectively introduced by setting $\lambda = 1$ and deleting Eq. (21b).

End and Initial Conditions

Conditions and data specified with respect to the current triad are, as a rule, easier to handle in intrinsic formulations. Specification of forces F_i and moments M_i at the ends poses no difficulty since they are subsets of the field variables. The same holds for kinematic conditions on the angular velocities α_i. The handling of linear velocity data $\boldsymbol{R}_{,t} = \boldsymbol{f}(t)$ is not as straightforward, since linear velocities are not included in the field variables. However, if $\boldsymbol{R}_{,t}$ is specified, then $\boldsymbol{R}_{,tt} = \boldsymbol{f}(t)_{,t} = \boldsymbol{A}(t)$ is also known at the end, assuming that it can be broken down into its components $A_i = f_i|_t$. Then the original equations of motion at the ends are to be used as end conditions:

$$F_i|_s = \rho A_i - q_i = \rho f_i|_t - q_i \tag{22}$$

The conditions involve derivatives of the forces and the loading. Note the reappearance of the uniform loading that dropped out of the field equations. This is as should be: uniform loading or acceleration should not cause any stresses unless there is interference with the motion, caused by the kinematic constraints.

Homogeneous end conditions can be regarded as intrinsic data and pose no problem. These include (among others) the classical cases.

$$\left.\begin{array}{ll} \text{Free end:} & \\ & F_i = M_i = 0 \\ \text{Hinged end:} & \\ & A_i = 0 \qquad M_i = 0 \\ \text{Clamped end:} & \\ & A_i = \alpha_{i=0} \end{array}\right\} \tag{23}$$

Extrinsic Conditions

In many practical situations the end conditions are specified in terms of fixed external directions. To use such data, the orientation of the triad at the end with respect to the external directions must be known at all times. This can be achieved by stepwise integrating the expressions for the triad at the end:

$$u_i(t+\Delta t) = \left[\delta_{ki} - e_{ijk}\alpha_j \Delta t - \tfrac{1}{2}(e_{ijk}\alpha_{j,t} - e_{ijm}e_{mrk}\alpha_j\alpha_r)(\Delta t)^2\right] u_k(t) + 0[(\Delta t)^3] \tag{24}$$

This equation should be appended to the list of field equations and used after each stepwise integration to recalculate the components of the end data. Although there is no theoretical difficulty in this approach, it makes the intrinsic formulation somewhat more cumbersome to apply and its use should be weighed against other possible formulations.

Initial Conditions

As in any dynamic problem, position, velocity, and state of deformation should be given at the initial time $t = t_0$. This requires, in general, that the beam be oriented at $t = t_0$ in the inertial frame and all its properties be known in this frame. From these, the initial values of the deformation and static variables can be directly calculated, so that the time integration process can be started. Satisfaction of the initial conditions also assures compatibility at the initial state. Together with Eqs. (5), compatibility is preserved for $t > t_0$. See example in Appendix B.

Summary of the Field Equations

The field equations for the fully intrinsic nonlinear dynamics of space beams consist of the transformed equations of linear and angular motion (14) and (20), equations of compatibility (5), and constitutive relations. Explicit forms for second covariant derivatives are given in Appendix A.

End conditions relate to the moments M_i, forces F_i, angular velocities α_i, and accelerations A_i. The latter represent data on the position or linear velocity at the end through Eq. (22). Extrinsic directional data at the end may require the use of Eq. (24).

The number of equations can be reduced as follows: F_1 and F_2 can be eliminated with the aid of the first two of the equations of angular motion and then F_3 and M_i can be eliminated with the aid of the constitutive relations. This leaves seven equations in the seven deformation variables (λ, κ_i, and α_i). The further elimination of the α_i (pure intrinsic dynamics) can be done in linear problems but is otherwise complicated. See the section on linear dynamics.

A detailed listing of the equations is given next. It is arranged in sequential order that can be followed in a direct time-integration scheme; for simplicity $m_1 = m_2 = 0$ and constant ρ were taken.

Transformed Equations of Motion

$$\rho\lambda_{,tt} = (F_3|_s)_{,s} - \kappa_2 F_1|_s + \kappa_1 F_2|_s + \lambda\left(\alpha_1^2 + \alpha_2^2\right) + (q_{3,s} - \kappa_2 q_1 + \kappa_1 q_2) \tag{25a}$$

$$\rho\left(\lambda^2\alpha_1\right)_{,t} = -\lambda[(F_2|_s)_{,s} - \kappa_1 F_3|_s + \kappa_3 F_1|_s] + \lambda^2\alpha_2\alpha_3 - \lambda(q_{2,s} - \kappa_1 q_3 + \kappa_3 q_1) \tag{25b}$$

$$\rho\left(\lambda^2\alpha_2\right)_{,t} = \lambda[(F_1|_s)_{,s} + \kappa_2 F_3|_s - \kappa_3 F_2|_s] - \lambda^2\alpha_1\alpha_3 + \lambda(q_{1,s} + \kappa_2 q_3 - \kappa_3 q_2) \tag{25c}$$

$$(I_1 + I_2)\alpha_{3,t} = M_{3,s} - \kappa_2 M_1 + \kappa_1 M_2 + (I_1 - I_2)\alpha_1\alpha_2 + m_3 \tag{25d}$$

Kinematics

$$\kappa_{1,t} = \alpha_{1,s} + \kappa_2\alpha_3 - \kappa_3\alpha_2 \tag{26a}$$

$$\kappa_{2,t} = \alpha_{2,s} + \kappa_3\alpha_1 - \kappa_1\alpha_3 \tag{26b}$$

$$\kappa_{3,t} = \alpha_{3,s} + \kappa_1\alpha_2 - \kappa_2\alpha_1 \tag{26c}$$

Constitutive Relations

$$F_3 = \tilde{F}_3(\lambda, \kappa_i, \ldots) \qquad [=C(\lambda - 1)] \qquad [=U_{,\lambda}] \tag{27a}$$

$$M_i = \tilde{M}_i(\kappa_j, \lambda, \ldots) \qquad \left\{=D_i\left[\kappa_i - \kappa_i^{(0)}\right]\right\} \qquad \left[=U_{,\kappa_i}\right] \tag{27b}$$

The expressions in brackets are for the linear BE case (ordinary approximate theory) and for the case where a strain energy density function U (per unit undeformed length) is available.

Auxiliary Equations

These express the quantities $F_i|_s$ in terms of F_3 and M_i:

$$F_3|_s = F_{3,s} + \lambda^{-1}[(\kappa_1 M_{1,s} + \kappa_2 M_{2,s}) + \kappa_3(\kappa_2 M_1 - \kappa_1 M_2)] \tag{28a}$$

$$F_1|_s = \kappa_2 F_3 - \left[\lambda^{-1}(M_{2,s} - \kappa_1 M_3 + \kappa_3 M_1)\right]_{,s} - \lambda^{-1}\kappa_3(M_{1,s} + \kappa_2 M_3 - \kappa_3 M_2) \tag{28b}$$

$$F_2|_s = -\kappa_1 F_3 + \left[\lambda^{-1}(M_{1,s} + \kappa_2 M_3 - \kappa_3 M_2)\right]_{,s} - \lambda^{-1}\kappa_3(M_{2,s} - \kappa_1 M_3 + \kappa_3 M_4) \tag{28c}$$

They are to be substituted back into the equations of motion but are listed separately for convenience.

Expressions for stepwise integration of the end triads are appended to the list of equations, as necessary (see the section on end conditions):

$$u_1(t+\Delta t) = \left[1 - \tfrac{1}{2}\left(\alpha_2^2 + \alpha_3^2\right)(\Delta t)^2\right]u_1(t) + \left[\alpha_3(\Delta t) + \tfrac{1}{2}(\alpha_{3,t} + \alpha_1\alpha_2)(\Delta t)^2\right]u_2(t) + \left[-\alpha_2(\Delta t) + \tfrac{1}{2}(-\alpha_{2,t} + \alpha_1\alpha_3)(\Delta t)^2\right]u_3(t) + 0[(\Delta t)^3] \tag{29}$$

where Δt is the time step. Expressions for u_2 and u_3 can be obtained from the preceding by cyclic permutation of the subscripts. The equations can be applied whenever the current directions of the triad are needed.

Numerical Integration of the Field Equations

A suitable method is that of stepwise integration in time, starting from the initial conditions. At each step, the list is advanced through a time step (Δt), proceeding from top to bottom. The quantities on the right side of the equations are taken from quantities already determined in the current step or, if not available, from the previous step. Partial or complete iterative loops can be performed. Where spatial derivatives are required, a discretization scheme can be used (finite differences of finite elements). More elaborate schemes exist.[8] In steady-state or quasistatic problems, a reversed order of solution may be chosen with spatial integrations.

Uniform Loads

A beam loading that is independent of s for all t, that is, $(1/\rho)\boldsymbol{q} = \bar{\boldsymbol{q}}(t)$, is a uniform loading. A common case is the weight of the beam itself. As explained before, these loads do not appear in the field equations and are reintroduced as end conditions whenever kinematic constraints are imposed, which interfere with the motion. To apply the loading term at the end, the orientation of $\boldsymbol{q}$ with respect to the triad must be known. There is no problem if the end is held against rotation ($\alpha_i = 0$) or if the angular velocities are specified at the end (such as in steady-state motion). In other cases of end constraint, where end values of α_i are not given, the stepwise integration scheme for the end triad must be used [Eq. (24 or 29)].

Planar, Fully Intrinsic Dynamics of Curved Beams

Let the motion take place in the $\boldsymbol{u}_1$–$\boldsymbol{u}_3$ plane. Set

$$\alpha_1 = \alpha_3 = \kappa_1 = \kappa_3 = M_1 = M_3 = m_1 = m_3$$

$$= F_2 = A_2 = q_2 = 0 \tag{30}$$

$$\alpha_2 = \alpha, \qquad \kappa_2 = \kappa, \qquad M_2 = M, \qquad m_2 = m \tag{31}$$

$$F_3 = N, \qquad F_1 = Q \tag{32}$$

Compatibility (5) reduces to the single equation $\kappa_{,t} = \alpha_{,s}$, which is identically satisfied by a function φ such that $\kappa = \varphi_{,s}$ and $\alpha = \varphi_{,t}$. It can be identified as the angle between $\boldsymbol{u}_3$ and the fixed x direction in the planar. Although the introduction of φ is not needed for intrinsic data, it is helpful for problems associated with external directions, which can be easily incorporated into the intrinsic-dynamic formulation. Taking, for simplicity, a constant ρ and $m = 0$, the equations reduce to

$$\rho(\lambda_{,tt} - \lambda\varphi_{,t}^2 = N_{,ss} - \varphi_{,s}^2 N - 2\varphi_{,s} Q_{,s}$$

$$- \varphi_{,ss} Q + q_{3,s} - \varphi_{,s} q_1 \tag{33}$$

$$\rho\lambda^{-1}\left(\lambda^2\varphi_{,t}\right)_{,t} = Q_{,ss} - \varphi_{,s}^2 Q + 2\varphi_{,s} N_{,s}$$

$$+ \varphi_{,ss} N + q_{1,s} + \varphi_{,s} q_3 \tag{34}$$

$$\lambda Q = -M_{,s} \tag{35}$$

A similar set of equations was developed by Libai[13] and was used for studying the large-amplitude flexural oscillations of straight bars. A later development that incorporates the effects of shearing deformations appears in Ref. 21. To add the shearing deformation effects, the following terms should be added to the right sides of Eqs. (33–35):

$$(\gamma q_1)_{,s} + \varphi_{,s}\gamma q_3 + \rho[\varphi_{,tt}\lambda\gamma + 2\varphi_{,t}(\lambda\gamma)_{,t}] \tag{36}$$

$$-(\gamma q_3)_{,s} + \gamma q_1 \varphi_{,s} - \rho\left[(\lambda\gamma)_{,tt} - \varphi_{,t}^2\lambda\gamma\right] \tag{37}$$

$$\gamma N + I_2\varphi_{,tt} \tag{38}$$

where γ is the shearing angle. Here, N and Q are taken to be normal and tangential to the (averaged) deformed cross section, and φ is its orientation. Appropriate constitutive relations should be added to complete the system.

Force, moment, and angular boundary conditions for Eqs. (33–35) present no problems. Kinematic constraints can be represented by specifying the accelerations A_1 and A_3 at the boundaries. These reduce to conditions on the derivatives of the forces as follows:

$$N_{,s} = \rho A_3 + \varphi_{,s} Q - q_3 \tag{39}$$

$$Q_{,s} = \rho A_1 - \varphi_{,s} N - q_1 \tag{40}$$

Uniform (gravity type) loading is given by $(1/\rho)q_1 = -g\cos\varphi$ and $(1/\rho)q_3 = -g\sin\varphi$. These drop out from the field equations but reappear in the boundary conditions when kinematic constraints are imposed.

The availability of φ as a field variable in the planar case makes it simple to pass from fixed external directions to the local deformed directions and vice versa. It is evident that all of the features of intrinsic beam dynamics are preserved in the planar case in a simplified form. In particular, it can take on strong nonlinearities, both material and geometrical. Many studies on planar dynamics exist, some of which are quite recent. (See cited literature and quoted references therein.)

Linear, Fully Intrinsic Dynamics of Space Beams

For sufficiently small incremental deformations and angular velocities from a known state of motion, the field equations can be linearized with respect to this state. As an example, consider linearization from the undeformed state, which should be a valid approximation for sufficiently small deformations, loads, and angular velocities. Putting $\varepsilon = \lambda - 1 =$ extensional strain, $k_i = \kappa_i - \kappa_i^{(0)} =$ curvature changes, substituting into the equations, and linearizing in F_i, M_i, k_i, ε, α_i, and q_i, a set of linear equations is arrived at:

$$\delta_{3k}\varepsilon_{,tt} - e_{3jk}\alpha_{j,t} = [(1/\rho)F_k|_s]|_s + [(1/\rho)q_k]|_s \tag{41}$$

$$(I_1 + I_2)\delta_{3k}\alpha_{3,t} - e_{3jk}F_j = M_k|_s + m_k \tag{42}$$

$$k_{i,t} = \alpha_i|_s \tag{43}$$

Here, the covariant operation has a linear form:

$$B|_t = B_{,t} \qquad \text{and} \qquad B_i|_s = B_{i,s} - e_{ijk}B_j\kappa_k^{(0)} \tag{44}$$

The constitutive relations are also linear, and so are the initial and end conditions.

Pure Linear Intrinsic Dynamics

These are dynamics that involve the pure deformation variables (ε and k_i) only. Using the formulas

$$k_{i,tt} = \alpha_i|_{ts} = \alpha_i|_{st} \tag{45}$$

which are valid for the linear case, α_i can be eliminated from the field equations. For simplicity, ρ and $I_1 + I_2$ are taken to constants. The detailed equations are

$$\rho\varepsilon_{,tt} = F_3|_{ss} + q_3|_s \tag{46}$$

$$(I_1 + I_2)k_{3,tt} = M_3|_{ss} + m_3|_s \tag{47}$$

$$\rho k_{1,tt} = (M_1|_{ss} + m_1|_s - q_2)|_{ss} \tag{48}$$

$$\rho k_{2,tt} = (M_2|_{ss} + m_2|_s + q_1)|_{ss} \tag{49}$$

To these, linear constitutive relations are added. Pure linear intrinsic dynamics are available for a general continuum, too (see Appendix B).

Discussion

Some of the merits and demerits of the dynamic intrinsic method are summarized in this section. These should be useful when possible approaches to a particular application are weighed.

Advantages

1) The equations are formulated in terms of direct deformation variables and can accommodate large deformations without undue complications.

2) Uniform loadings and (translational) accelerations drop out of the equations. Stresses arise from nonuniformities and from geometrical constraints. In problems involving large-scale motion that is interwoven with small scale nonlinear deformations, the automatic canceling out of the large-scale motion is facilitated.

3) The method utilizes the current geometry as reference and can accommodate loadings and end conditions that are specified with respect to it. This includes pressure, drag, some specified end loads, and also the classical conditions of free end, clamped end, and simply supported end.

4) Some specialized theories are easily formulated. Examples are inextensional theory ($\lambda = 1$) and steady-state rotation ($\kappa_{i,t} = \lambda_{,t} = \alpha_{i,t} = 0$).

5) Constitutive relations are applied directly to the field variables, and inelastic and time-dependent properties are easily introduced.

6) The equations are presented in a time-rate form, which is convenient for time-integration schemes.

Drawbacks

1) The method is specifically oriented towards solving dynamic problems. Although static problems can be accommodated, they can be more easily solved by other means.

2) If the current position is an important objective of the analysis, then displacement formulations or mixed formulations are more advantageous.

3) Data with respect to fixed directions in space cannot be handled easily (the planar case is an exception).

4) The method is strain-curvature oriented and, essentially, uses first and second displacement derivatives for its field variables. Hence, specific care regarding possible discontinuities must be exercised in the construction of its numerical algorithms. For example, a discontinuity in a curvature implies a second-derivative discontinuity in displacement formulations, whereas in intrinsic methods it implies a discontinuity in the field variable itself.

5) There are no well-established solution routines or variational techniques available for applicable cases. These still await development.

Appendix A: Space and Time Derivatives on a Beam

For compactness, it is useful to regard the equation of the position vector $\boldsymbol{R}(s,t)$ of the centroidal line of a space beam for all s and t as that of a surface in three-dimensional space with coordinate lines $\xi^\alpha(\xi^1 \equiv s, \xi^2 \equiv t)$. Coordinate lines $\xi^2 = C$ denote the shape of the beam at a given time, whereas the lines $\xi^1 = C$ denote the trajectory of a material point with initial positions. The rotational velocities of the unit triad on $\boldsymbol{R}$ are denoted by $\omega_{\alpha i}(\alpha = 1, 2)$ with $\omega_{1i} = \kappa_i$ and $\omega_{2i} = \alpha_i$. Here, Greek indices relate to the surface and Latin indices relate to the triad $\boldsymbol{u}_i$.

For any three-dimensional vector $\boldsymbol{B}(s,t)$ defined on the beam, the derivatives can be expressed in terms of the triad as follows:

$$\boldsymbol{B}_{,\alpha} = B_i|_\alpha \boldsymbol{u}_i; \qquad \boldsymbol{B}_{,\alpha\beta} = B_i|_{\alpha\beta}\boldsymbol{u}_i \tag{A1}$$

It follows that

$$\boldsymbol{u}_{i,\alpha} = -e_{ijk}\omega_{\alpha j}\boldsymbol{u}_k \tag{A2}$$

$$\boldsymbol{u}_{i,\alpha\beta} = (-e_{ijk}\omega_{\alpha j,\beta} + e_{pij}e_{prk}\omega_{\alpha j}\omega_{\beta r})\boldsymbol{u}_k \tag{A3}$$

$$B_i|_\alpha = B_{i,\alpha} + e_{ijk}\omega_{\alpha j}B_k \tag{A4}$$

$$B_i|_{\alpha\beta} = B_{i,\alpha\beta} + e_{ijk}(\omega_{\alpha j}B_{k,\beta} + \omega_{\beta j}B_{k,\alpha}) + (e_{ijk}\omega_{\alpha j,\beta} - e_{pjk}e_{pri}\omega_{\alpha j}\omega_{\beta r})B_k \tag{A5}$$

The space–time compatibility equations assure the symmetry of the last term:

$$\omega_{\alpha i,\beta} - \omega_{\beta i,\alpha} = e_{ijk}\omega_{\alpha j}\omega_{\beta k} \tag{A6}$$

Appendix B: Intrinsic Dynamics of a Continuum

The equations of motion of small deformations of a continuous medium with a constant mass density ρ are

$$F_j = \tau_j^i\big|_i + P_j = \rho u_{j,tt} \tag{B1}$$

where $\tau_j^i|_i$ is the mixed stress tensor, P_j is the loading, u_j are the displacements, and F_j is the force unbalance that is introduced here for convenience. The strain tensor is given by

$$2e_{ij} = u_i|_j + u_j|_i \tag{B2}$$

Differentiation and elimination of the u_j yield the intrinsic equations of motion:

$$2\rho e_{jk,tt} = F_j|_k + F_k|_j \tag{B3}$$

$$= \tau_j^i\big|_{ik} + \tau_k^i\big|_{ij} + P_j|_k + P_k|_j \tag{B4}$$

Linear constitutive relations or algorithms relate the stresses to the strains and possibly their time derivatives. The simplest relation is

$$\tau^{ij} = C^{ijkL}e_{kL} \tag{B5}$$

Substitution into the preceding yields six equations in the e_{ij} and their space and time derivatives.

Boundary and Initial Conditions

Stress boundary conditions pose no problem. Kinematic boundary conditions require the specification of $u_{i,t} = f_i(t)$ on the boundary ∂V. Substitution into Eq. (B1) yields a condition on the stress gradients

$$\tau_j^i\big|_i = f_{j,t} - P_j \tag{B6}$$

The ease of implementation of the kinematic boundary condition constitutes an important advantage of the dynamic intrinsic method over corresponding static intrinsic formulations.

Initial conditions at $t = t_0$ consist of specifying u_i and $u_{i,t}$ at $t = t_0$. From these, e_{ij} and $e_{ij,t}$ can be calculated directly by spatial differentiation.

Compatibility

Let $L_i(e_{jk}) = 0\,(i = 1, \ldots, 6)$ be the six compatibility equations of linear continuum mechanics. The formal analogy between Eqs. (B2) and (B3) (with F_j replacing u_j) assures that elimination of the F_j from Eq. (B3) would result in

$$L_i(e_{jk,tt}) = 0 \qquad (i = 1, \ldots, 6) \tag{B7}$$

The initial conditions assure that $L_i(e_{jk}) = 0$ and $L_i(e_{jk,t}) = 0$ at $t = t_0$. It follows that compatibility will be satisfied for all $t > t_0$. This implies that the system of intrinsic dynamics satisfies the compatibility equations automatically.

Regarding angular motion, the antisymmetric part of the $F_j|_k$ tensor is related to angular accelerations. If position is of interest, it can be used to calculate the motion. Otherwise, it is of no interest in linear analysis.

Nonlinear Analysis

Here, angular velocities contribute to the translational accelerations and must be included. Let the velocity gradient d_{ij} be defined by

$$d_{ij} = \tfrac{1}{2}g_{ij,t} - E_{ijk}\alpha^k \tag{B8}$$

where E_{ijk} is the tensorial permutation symbol, α^k are the angular velocities, and g_{ij} is the current metric tensor. The equations reduce, in this case, to the following closed system.

Motion:

$$d_{ij,t} = \left[(1/\rho)\left(\tau_j^k\big|_k + P_j\right)\right]\Big|_i + d_{ik}d_j^k \tag{B9}$$

Kinematics:

$$g_{ij,t} = d_{ij} + d_{ji} \tag{B10}$$

Constitutive:

$$\tau^{ij} = f^{ij}(g_{kL}\ldots)\left(= C^{ijkL}e_{kL}\right) \tag{B11}$$

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References

[1] Rosen, A., "Structural and Dynamic Behavior of Pretwisted Rods and Beams," *Applied Mechanics Reviews*, Vol. 44, No. 12, Pt. 1, 1991, pp. 483–515.

[2] Hodges, D. H., "Nonlinear Beam Kinematics for Small Strains and Finite Relations," *Vertica*, Vol. 11, No. 3, 1987, pp. 573–589.

[3] Love, A. E., *A Treatise on the Mathematical Theory of Elasticity*, 4th ed., Dover, New York, 1927, Chaps. 16–19.

[4] Antman, S. S., "The Theory of Rods," *Mechanics of Solids*, Vol. II, edited by C. Truesdell, Springer–Verlag, Berlin, 1984.

[5] Reissner, E., "On Finite Deformation of Space-Curved Beams," *Journal of Applied Mathematics and Physics*, Vol. 32, No. 6, 1981, pp. 734–744.

[6] Danielson, D. A., and Hodges, D. H., "A Beam Theory for Large Global Rotation, Moderate Local Rotation, and Small Strain," *Journal of Applied Mechanics*, Vol. 55, March 1988, pp. 179–184.

[7]Simo, J. C., "A Finite Strain Beam Formulation. The Three-Dimensional Dynamic Problem. Part I," *Computer Methods in Applied Mechanics and Engineering*, Vol. 49, No. 1, 1985, pp. 55–70.

[8]Simo, J. C., and Vu-Quoc, L., "On the Dynamics in Space of Rods Undergoing Large Motions—A Geometrically Exact Approach," *Computer Methods in Applied Mechanics and Engineering*, Vol. 66, No. 2, 1988, pp. 125–161.

[9]Hodges, D. H., "A Mixed Variational Formulation Based on Exact Intrinsic Equations for Dynamics of Moving Beams," *International Journal of Solids and Structures*, Vol. 26, No. 11, 1990, pp. 1253–1273.

[10]McDonough, T. B., "Formulation of the Global Equations of Motion of a Deformable Body," *AIAA Journal*, Vol. 14, No. 5, 1976, pp. 656–660.

[11]Canavin, J. R., and Likins, P. W., "Floating Reference Frames for Flexible Spacecraft," *Journal of Spacecraft and Rockets*, Vol. 14, No. 2, 1977, pp. 724–732.

[12]Laskin, R. A., Likins, P. W., and Longman, R. W., "Dynamical Equations of a Free-Free Beam Subject to Large Overall Motions," *Journal of the Astronautical Sciences*, Vol. 31, No. 4, 1983, pp. 507–528.

[13]Davis, J. H., and Hirschorn, R. M., "A Model for the Embedded Tendon Control of a Slender Three-Dimensional Flexible Robot Link," *Dynamics and Control*, Vol. 4, April 1994, pp. 185–208.

[14]Libai, A., "Intrinsic Equations for the Nonlinear Dynamics of Space Beams," AIAA Paper 94-1676, April 1994.

[15]Green, A. E., and Laws, N., "A General Theory of Rods," *Proceedings of the Royal Society of London*, Series A, Vol. 293, 1966, pp. 145–155.

[16]Naghdi, P. M., and Rubin, M. B., "Constrained Theories of Rods," *Journal of Elasticity*, Vol. 14, No. 4, 1984, pp. 343–361.

[17]Libai, A., "On the Nonlinear Elastokinetics of Shells and Beams," *Journal of the Aerospace Sciences*, Vol. 29, No. 10, 1962, pp. 1190–1195, 1209.

[18]Libai, A., "On the Nonlinear Intrinsic Dynamics of Doubly Curved Shells," *Journal of Applied Mechanics*, Vol. 48, Dec. 1981, pp. 909–914.

[19]Libai, A., "Nonlinear Shell Dynamics—Intrinsic and Semi-Intrinsic Approaches," *Journal of Applied Mechanics*, Vol. 50, Sept. 1983, pp. 531–536.

[20]Libai, A., and Simmonds, J. G., "Nonlinear Elastic Shell Theory," *Advances in Applied Mechanics*, Vol. 23, Academic, New York, 1983, pp. 271–371 (Sec. III.J.3).

[21]Libai, A., and Simmonds, J. G., *The Nonlinear Theory of Elastic Shells*, Academic, San Diego, CA, 1988, pp. 38, 112, 113, 291–293.

[22]Simo, J. C., and Vu-Quoc, L., "On the Dynamics of Flexible Beams Under Large Overall Motions—The Planar Case: Part I," *Journal of Applied Mechanics*, Vol. 53, Dec. 1986, pp. 849–854; also Pt. II, pp. 855–863.